Evolutionary Graph Theory

Fixation times on directed graphs

David Brewster

Harvard University

Joint work w/ Martin Nowak and Josef Tkadlec

Evolutionary graph theory

How does stuff propagate through networks?

Maximizing the spread of influence through a social network
D Kempe, J Kleinberg, É Tardos - … of the ninth ACM SIGKDD international …, 2003 - dl.acm.org
Models for the processes by which ideas and influence propagate through a social network have been studied in a number of domains, including the diffusion of medical and technological innovations, the sudden and widespread adoption of various strategies in game-theoretic settings, and the effects of" word of mouth" in the promotion of new products. Recently, motivated by the design of viral marketing strategies, Domingos and Richardson posed a fundamental algorithmic problem for such social network processes: if we can try to …
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[HTML] Complex networks: Structure and dynamics
S Boccaletti, V Latora, Y Moreno, M Chavez… - Physics reports, 2006 - Elsevier
Coupled biological and chemical systems, neural networks, social interacting species, the Internet and the World Wide Web, are only a few examples of systems composed by a large …
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[HTML] **Evolutionary games** on **graphs**
G Szabó, G Fath - Physics reports, 2007 - Elsevier
Game theory is one of the key paradigms behind many scientific disciplines from biology to behavioral sciences to economics. In its **evolutionary** form and especially when the …
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Statistical physics of **social dynamics**
C Castellano, S Fortunato, V Loreto - Reviews of modern **physics**, 2009 - APS
… best to mention relevant **social** science literature and highlight connections to it, the main focus of this work remains a description of the **statistical physics** approach to **social dynamics**. …
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[HTML] **Statistical physics** of human cooperation
M Perc, JJ Jordan, DG Rand, Z Wang, S Boccaletti… - **Physics** Reports, 2017 - Elsevier
… the relevance of **physics** in all of this. Methods of **statistical physics** have recently been … **Statistical physics** of **social dynamics** [13], of evolutionary games in structured populations […
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- coronavirus among humans
- influence (opinion, gossip, fake news) on social media
- genetic mutation in a population of individual organisms

Model: Moran process on a graph

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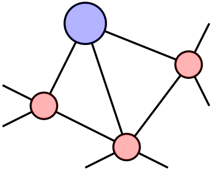

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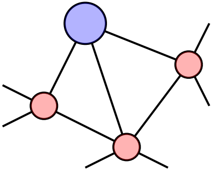

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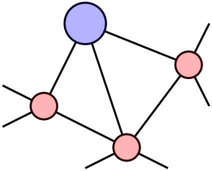

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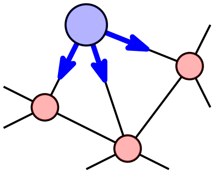

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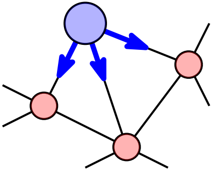

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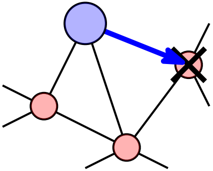

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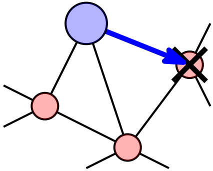

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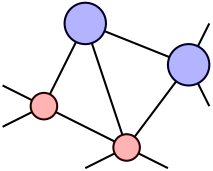

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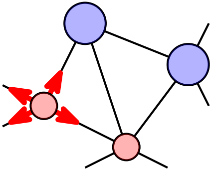

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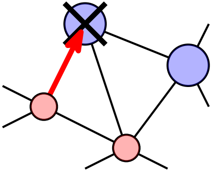

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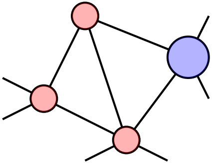

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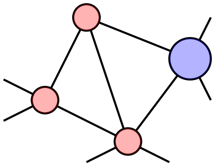

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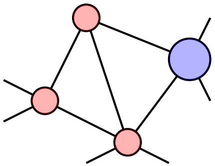
Some features of the Moran process



	fitness
mutant	r
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1. It is stochastic (random).
2. In some steps, nothing happens.
3. Nodes can toggle back and forth (more opinions than gossip).
4. Eventually, all nodes become the same type (no mutation).
5. Variants exist (e.g. death-Birth updating).

Quantities of interest:

1. Fixation probability $\text{fp}^r(G)$: Average probability that, starting from a single node, mutants spread to all sites.
2. Fixation time $T^r(G)$: Average time until one type wins.
   - Measured in steps or better in generations.

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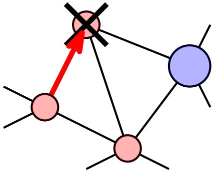

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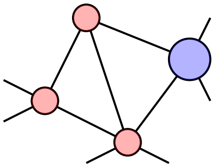

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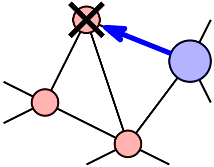

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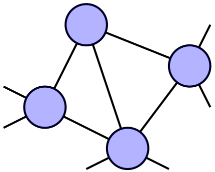

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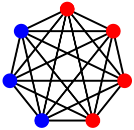
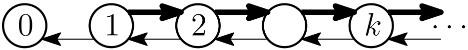
Special case: Complete graph K_n and $r > 1$



$F = kr + (n-k)$

$$\left.\begin{aligned} p_k^+ &= \frac{kr}{F} \cdot \frac{n-k}{n-1} \\ p_k^- &= \frac{n-k}{F} \cdot \frac{k}{n-1} \end{aligned}\right\} \quad \frac{p_k^+}{p_k^-} = r$$

It turns out that we are always r-times more likely to gain than to lose a mutant. Thus the process can be mapped to a 1-dimensional random walk, with a constant forward bias r.



Claim. $\text{fp}^r(K_n) = \frac{1-1/r}{1-1/r^n} \to_{n\to\infty} \boxed{1 - 1/r}$.

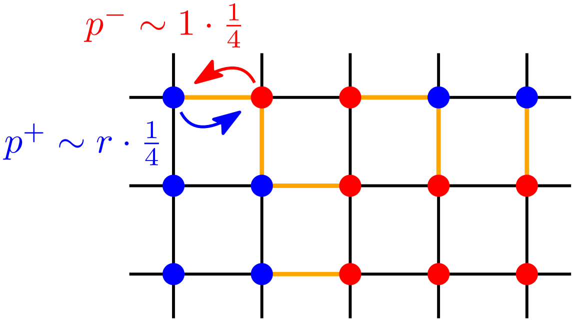
(Intuition. Let $x = 1 - \text{fp}^r(K_n)$. Then $x = \frac{1}{r+1} \cdot 1 + \frac{r}{r+1} \cdot x^2$.)

Special case: Regular graphs R_n

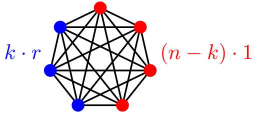
Claim (Isothermal theorem, '05). For any regular graph we have

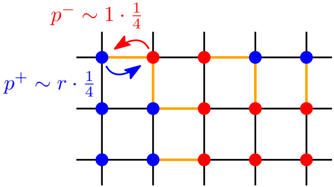
$$\mathrm{fp}^r(R_n) = \mathrm{fp}^r(K_n).$$

Proof. The same mapping works! We say that an edge is active if its endpoints are of different types. Each active edge is r-times more likely to be used in gaining rather than losing a mutant.

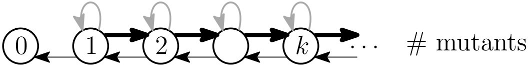


But #steps on regular graphs differ





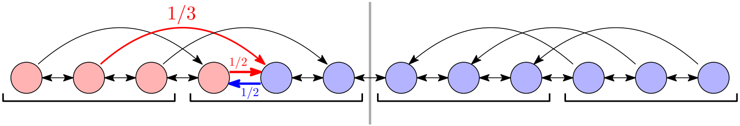
Intuition. If a of E edges are active, then, on average, roughly one in every E/a steps is active.

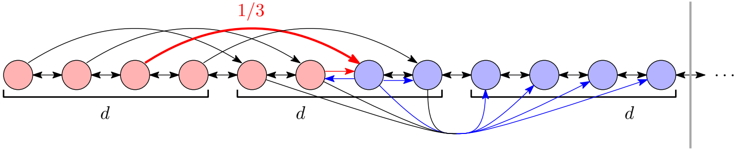


K_n: #edges $\sim n^2$, #active edges $\sim k(n-k)$
$\rightarrow$ #steps for $K_n \sim c \cdot n + \sum_k \frac{n^2}{k(n-k)} \sim \Theta(n \log n)$

Sq_n: #edges $\sim 4n$, #active edges $\in (\sqrt{k}, 4k)$
$\rightarrow$ #steps for Sq_n is $\mathcal{O}(n\sqrt{n})$ and $\Omega(n \log n)$.

Simulations can be slow on directed graphs





Directed graphs

[B-Nowak-Tkadlec '23+]

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3. For some graphs (even undirected), increasing mutant advantage can result in longer fixation times.

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5. If graph is *balanced*, fixation occurs quickly for any $r \geq 1$.

Balanced graphs

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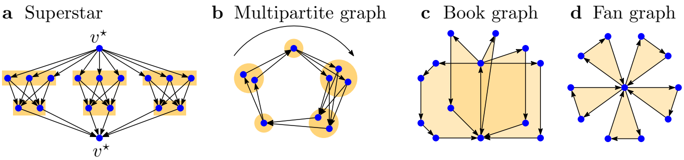

Games on directed graphs

- Directed graph $G = (V, E)$
- death-Birth updating
- Given death node $v \in V$, fitness of $u \in \Gamma^{-}(v)$ is $1 - w + wP$ where P is total payoff from playing games with nodes in $\Gamma^{-}(v)$

	Mutant	Resident
Mutant	r	r
Resident	1	1

	Cooperate	Defect
Cooperate	$b - c$	$-c$
Defect	b	0

for $r > 1$ and $b > c > 0$.

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Questions?